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The Progressive Nature of Learning in Mathematics[1]

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William A. Brownell was a towering figure in mathematics education in the early twentieth century. We have chosen to share his "meaning theory" of mathematics by reprinting his 1944 article "The Progressive Nature of Learning in Mathematics." In this article, Brownell criticizes behavioral theories of learning and argues that teaching mathematics from a behaviorist perspective results in "pseudo-learning, memorization, and superficial, empty verbalization."

The ideas expressed by Brownell remain every bit as relevant to mathematics teachers today as they did in the 1940s. These include the importance of focusing on the processes of learning, not just the products; teaching fewer topics in more depth; how an ineffective use of drill and practice can interfere with the development of meaningful learning; and the importance of improving our assessment and evaluation techniques.

Readers may be surprised at the modern ideas expressed by Brownell more than sixty years ago and will find numerous parallels to current discussions about conceptual understanding, problem solving, and the provision of quality mathematics education for all students.

There is a close relation between our teaching procedures and our conception of the learning process. As the first step in teaching we more or less carefully determine our objectives. Then, to help our pupils attain these objectives, we select our explanations, our types of practice materials, and our applications largely in the light of our theory of learning. It follows therefore that his view of learning is a critical part of every teacher's professional equipment.

LEARNING AS CONNECTION-FORMING

The conception of learning which has prevailed in American education for more than a quarter-century—less so now than formerly—is part of the psychological system known as connectionism. According to this view, all learning consists in the addition, the elimination, and the organization of connections—this, and nothing else. These connections are formed, or broken, or organized, between situations and responses. The process of teaching, then, comprises the following steps:

1. Identify for the learner the stimuli (or the situation) to which he is to react,
2. Identify the reaction (or response) which he is to make,
3. Have the learner make this response to the situation under conditions which reward success and which punish failure,

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4. Repeat step (3) until the connection has been firmly established.

The connectionistic view of learning has not, in my opinion, been very helpful to teachers.[2] Advocates of connectionism could hardly accept my evaluation. If they were to concede that their view of learning has not always produced the best results, they would insist that the deficiency lies, not in the theory, but in the user. They would say, as many of them have said, that the theory is sound and adequate, but that it has been misinterpreted and misapplied.[3]

In this paper I want to consider with you four weaknesses in classroom instruction in mathematics. (They are by no means confined to the teaching of mathematics.) These weaknesses persist after thirty years and more of connectionism. Whether they are still with us because of the connectionistic view of learning or in spite of it, it is impossible to say. It is however possible to show how this view of learning seems to support, even to demand, the malpractices which I shall discuss. It is for this reason that the connectionistic view of learning will come in for unfavorable comment.

I am well aware that I am talking, not to professional psychologists, but to teachers of mathematics. Indeed, it is precisely because I *am* talking to teachers of mathematics that I speak as I shall. Perhaps no other subject in the curriculum so much as mathematics has suffered from the general application or the general misapplication of connectionistic theory. But my remarks will not all be negative. On the contrary, as the subject of the paper implies, I shall try to substitute positive notions and shall try to sketch a different view of learning which may be more useful to teachers.

FOUR INSTRUCTIONAL WEAKNESSES IN MATHEMATICS

The four instructional weaknesses to which connectionistic theory has contributed directly or indirectly are:

1. Our attention as teachers is directed away from the processes by which children learn, while we are over-concerned about the product of learning,
2. Our pace of instruction is too rapid, while we fail to give learners the aids they need to forestall or surmount difficulty,
3. We provide the wrong kinds of practice to promote sound learning,
4. Our evaluation of error and our treatment of error are superficial.[4]

1. Process vs. product.—I have said that connectionistic theory leads us to neglect the processes by which children learn. This is so because, unlike the *product* of learning, the *process* of learning seems scarcely worthy of attention. The reasoning is somewhat like this: all learning is the formation of connections; or, the process of learning *is* the making of connections. Connections all being basically alike, the process is the same for all learning. Hence, we need only to make sure that the right connections are established; they will then necessarily lead to correct responses. We know what the correct responses are; we identify them for learners, along with the appropriate stimuli, and we provide practice until the desired connections between the two are formed. The process of learning, which is to say, connection-forming, takes care of itself. From all this we come to teach by telling children or showing children what to do and then by seeing that they do it.

> Mathematics has suffered from the general application or misapplication of connectionistic theory

We tell children that 2 and 5 make 7; we show them how to divide one fraction by another; we give them the rules governing signs in algebraic operations; we furnish them the facts of the Pythagorean Theorem. Practice follows to establish the connections. When our pupils demonstrate that they have the desired connections by producing the correct responses, the teaching job is done.

Now, more is the pity, some children try to learn mathematics according to this simple pattern. I shall have more to say about these children at a later point. It is pertinent here, however, to note the attitudes which such children develop toward mathematics. The correct answer is their sole consideration. Let them, by no matter what curious manipulation of symbols, arrive at an answer which agrees with that of teacher or textbook, and they feel that they have met all requirements of the situation. Change in the slightest degree the conditions in which the mathematics occurs, and they are helpless. Challenge an answer even when it is correct, and they have no way to prove it. To tell them that mathematics, whether it be arithmetic or algebra or geometry, is a system of logical relations is to speak in a foreign language.

But a large percent of children do not learn mathematics in this simple, blind way, even when the teaching might seem to encourage such learning. Instead, they try somehow to put sense into what they learn. They may be told—and told time and again—that 2 and 5 make 7; but they forget it. When the forgetting becomes too embarrassing, they try something beside memorization. They turn 2 and 5 around to make 5 and 2, which for some reason they may know better; or they count one number onto the other as a base; or they see 2 and 5 as the same as 3 and 4, with a sum of 7.

You are probably all familiar with the girl reported by Stephenson, a girl who has many mathematical relatives in every community in which I have lived. This girl found verbal problems too much for her; or, they were too much for her until she devised some simple rules: when the problem contains several numbers, you add; when it contains two long numbers, you subtract; when the larger of two numbers exactly contains the smaller, you divide; otherwise, you multiply. This girl, like her many previously mentioned relatives, has been held up as a conspicuous example of stupidity. I cannot agree. I believe, instead, that she showed an extraordinary degree of originality and resourcefulness. As a matter of fact, barring computational errors, she would, by her procedures, get the correct answers for the majority of problems in texts for the lower and intermediate grades. And if she got the wrong answers? Well, so did the other children who had different "rules." It just happens that the processes used by this girl are not mathematical, and she was supposed to be working in the area of mathematics. But mathematics as such meant nothing to her; she had not learned the tricks of the trade; so, she invented some of her own.

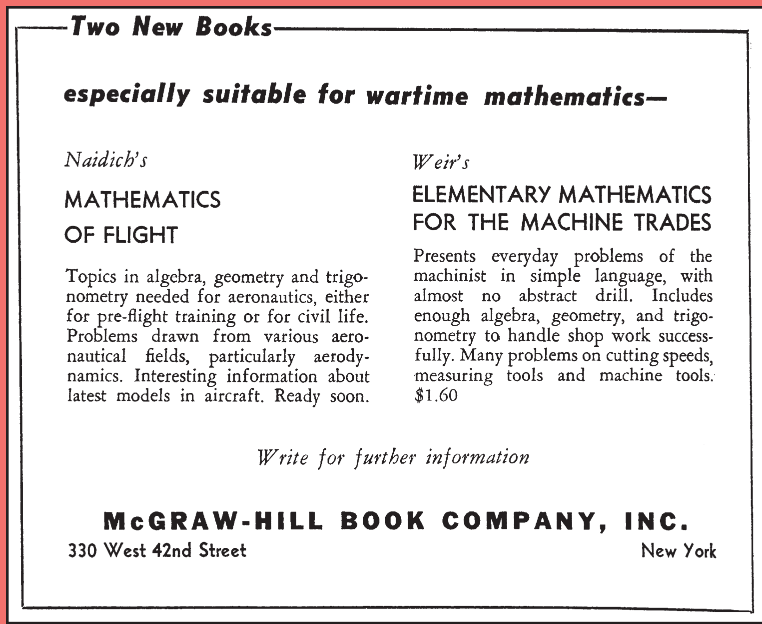
If she thinks of learning purely as the formation of connections, she is apt to oversimplify the learning task

What I have been doing, you will have recognized, is to illustrate negatively the importance of *process* in learning. Let us now approach the matter positively. Consider the example 42 + 27. There is not just one way to find the answer, as we sometimes naively assume. There are almost numberless ways:

a. One may count out 42 separate objects, lay them aside, count out 27 more similar objects, lay them aside in a separate group; and then find the total by counting all the objects by 1's, starting with 1.
b. One may use objects as in a, getting two groups of 42 and 27, and again count by 1's, but start with 43 or 28 instead of with 1.
c. and d. One may use objects as in a and b, but count by larger units than 1's, as by 2's or 5's.
e. Still using objects, one may set up 42 as four groups of 10 objects each, and 2 over, and 27 as two groups of 10, with 7 over; he may then count the tens, getting 6, and the ones, getting 9, to yield the total of 69.
f. to j. One may use any of the first five methods, substituting marks or pictorial symbols for actual objects.
k. One may count the 27 onto the 42 abstractly, or the 42 onto the 27.
l. and m. One may copy the abstract numbers and get subtotals in the two columns by counting abstractly or with marks.
n. One may know the separate combinations and add directly: 2 and 7 are 9; 4 and 2 are 6; total, 69, without knowing anything about the composition of the numbers dealt with—a purely mechanical stunt.
o. One may proceed as in n, but be fully aware of the nature of numbers and of the process of addition.
p. to n. One may do any one of the many things which this audience would report as their processes, such as: (1) direct and immediate apprehension of the total; (2) adding 20 to 42, and then adding the remaining 7; (3) adding 40 and 20, adding 2 and 7, and then combining; (4) adding from the left, with a preliminary glance to make sure that no carrying is involved; and so on, and so on.

All these procedures, and others not here catalogued, are entirely legitimate. All of them, except the last few, may be found in actual use in classrooms in which the procedure for adding such numbers as 42 and 27 is being taught. The teacher knows the *product* she is seeking to attain, namely, skill in adding two-place numbers without carrying. If she thinks of learning purely as the formation of connections, she is apt to oversimplify the learning task. She will tend to show her pupils how to add the digits in the two columns and where to place the partial sums; and then she will rely on practice to establish the needed connections.

I have tried to show that identification of the learning process with the formation of connections, however valid for ultimate psychological and neurological theory, is not useful to teachers. Teaching is the guidance of learning. We can guide learning most effectively when we know what the learners assigned to us really do in the face of their learning tasks. In a word, we as teachers can be helpful in guidance to the degree to which we know our pupils' processes. I do not mean that the product of those processes is no concern of ours; but I do mean that processes are of at least equal importance with products. The teacher who knows the product which is to be finally achieved, but who also knows how to discover, evaluate, and direct the processes of her pupils as they approach this goal—that teacher is probably a good teacher. Moreover, thinking of learning as the formation of connections would not make her a better teacher.



Sample ad, *Mathematics Teacher* 36 (October 1943)

2. Over-rapid instruction.—So much for the first objection to the connectionistic view of learning: it takes us as teachers away from our main stock in trade, namely, the processes by which children learn. The first objection is closely related to the second: it tends to make us hurry unduly the pace of instruction and it discourages us from supplying to children temporary aids and procedures which they need for sound learning. In a word, we are led to think that children can complete their learning at a single jump.

Without reciting them again, let me recall the list of processes by which one may find the sum for 42 and 27. These processes were described in the earlier place only to establish the existence of various possible processes. Let me cite them again here for another purpose. The processes were arranged by plan, in a roughly ascending order of complexity, maturity, and abstractness. It is surely obvious that the child who counts by 1's 42 objects, then 27, and finally the total of 69 is acting more simply and concretely and less maturely than does the adult who apprehends at a glance the sum of the two abstract numbers. And the other processes listed can be posted at intermediate points in the scale of maturity and abstractness. Any given child may be at any point in this scale. Furthermore, his degree of mastery of his process, whatever it is, may vary from inexpertness to striking proficiency. Indeed, a child whose procedure (counting by 1's, for example) is very low in the scale of maturity may outscore in rate and accuracy another child whose procedure is at a higher point in the scale. Failure to note this fact is one of the penalties we pay for neglecting process in favor of product only, and it constitutes a large source of error in the evaluation of learning.

We have come to accept the typical curves of learning as picturing all that goes on in learning. The accuracy curve, for example, mounts rapidly at first and then slower and slower. The implication is that the child is getting the desired connection established just about as is portrayed in the curve. But when we examine into the behavior of the child, we find the situation to be much more complex than this. He may practice a given procedure for a time and then desert it for another, and this for another, and that for still another. A more valid picture of his learning, this time plotted in terms of process, would look something like a series of steps, each successive one somewhat higher in the scale of maturity than the preceding one. In a word, the learner progresses by traversing a series of stages in thinking. Each stage serves its purpose for a time, but is superseded by a more advanced stage. As each stage is abandoned for the next, the earlier stage is not forgotten or gone—its pattern is not eradicated from the nervous system. Instead, the older procedure is overlaid by another, and the old neural pattern remains for use if for any reason the more recently acquired procedure does not function smoothly.

This description of learning, while it may be consistent with connectionism, is certainly not suggested by connectionism. On the contrary, the connectionistic view of learning leads us to give the child at the outset the form of response which we want him ultimately to have. And we are inclined to do this quite without regard to his attained stage of thinking when we present the new learning task. At best, the result is pseudo-learning, memorization, and superficial, empty verbalization.

Let me illustrate what I mean. I once asked a third-grade teacher to send me for interviews the three poorest arithmetic pupils in her class. George was one of the three. He was described as an almost certain failure. After some preliminary exercises, I put before George the three digits 8, 7, and 9 arranged for column addition, and asked him to find the sum. Listen to him: "8 and 7 are 15, and 9 are . . . I don't know." Then, in the reverse direction: "9 and 7 are 16, and 8 are . . . I don't know." After a pause George looked at me and asked, "May I count?" I told George he might, and promptly he mumbled, "8 and 7 are 15, 16, 17, . . . 24." When I asked him to add upward, he produced the correct answer with equal quickness by counting the 8 to the sum of 9 and 7. Obviously, George did not understand bridging of the decades, save as this could be done by counting.

But why did George ask if he might count? Clearly because his teacher had told him that he must not count, with what penalties I do not know. George obeyed her, and the reward for his obedience was failure. Parenthetically may I say that one of the three *best* pupils sent from the class for interviews counted not only to bridge, but counted to find the sum of each pair of digits. This last child, singled out for praise, was really at a lower stage of thinking than was George; but the teacher did not know it. She had asked these children to perform at a level or stage of thinking of which they were then incapable, and they had reacted according to their natures.

To return to George and to repeat what I have already said—this lad was at a very immature level with respect to bridging: he had to count if he was to bridge at all. I said to George, "George, how many are 8 and 7?" He said at once, "15." I asked, "How many more does it take to make 20?" He said promptly, "5"; and then after an instant his face lit up and he said, "And 4 more are 24!" Then, "Will it work like that every time?" I asked him to add the same numbers upward, and he said, "9 and 7 are 16; . . . and 4 are 20, and 4 are 24! Can I do that with all of them?" I wrote several examples which called for bridging 20, and he solved them quickly, understandingly, and triumphantly by his new method.

What had I done? Well, I had helped George to the next higher stage in thinking, a stage for which he was ready, but a stage which he had not discovered for himself and which he was unlikely to find, so long as instruction consisted only in telling what he must not do.

Should George have been left at the stage to which I had helped him, always to bridge 20 or some higher decade by splitting a number? No; he should have been led next to understand the principle of adding by endings, and still later to think immediately and directly of the total of the last partial sum and the last digit. Sound learning required that he traverse these intermediate steps; he was getting nowhere by his own devices. Unfortunately circumstances prevented my helping him discover the next steps, and I fear that his teacher may not have provided the needed assistance either.

At best, the result is pseudo-learning, memorization, and superficial, empty verbalization

I have spent all this time on George because his case illustrates so clearly the dangers of too rapid a pace of instruction. It illustrates too the need for temporary or intermediate processes, methods, and devices which are now commonly kept from children to their detriment. These temporary aids have a bad reputation in education. I am sure you have often heard the dictum: "Never form a habit which must later be broken." The addition of the qualifying phrase "Other things being equal," whatever its intent, does little to soften the ban against their use. Indeed, so fully are school officials and teachers persuaded of the evils of these aids that they are to be found in few textbooks, and there in insufficient amount. I do not mean however to imply that these aids are totally unknown to the classroom. Good teachers use them—sometimes openly, perhaps more often when the supervisor or principal is not likely to appear.

So long as we think of learning as a simple, straight-line development, the warning against temporary aids makes sense. These aids seem to contribute little; they increase the number of things to be learned; they tend to be retained after they have long outlived their usefulness (if any). But we must cease to think of learning, and certainly of learning in mathematics, in this manner. We do not seek merely to develop a few mechanical skills always to be used precisely as they were learned. The purpose of mathematics, whether in the elementary school or in the high school, goes far beyond the establishment of mechanical skills. The ideas, principles, generalizations, and relationships which are taught, *as well as the skills,* are intended for purposes outside themselves and for use in situations quite unlike those in which they are learned. We teach quantitative mathematics, for example, as a system of thinking by which to manage and control number and quantity, not alone as presented in textbook problems or as presented in the classroom, but however and wherever and whenever presented. In a word, we strive to teach understandings. When the goal of understanding is accepted, the function of temporary aids is seen in its correct perspective. Such aids contribute meanings when meanings are needed; and the more meanings, the deeper the understanding, and the greater the chances of successful transfer to new and unfamiliar situations.

In making a case for temporary aids I may have failed to indicate all that should be included in this category. My illustrations have all been what may be called "lower-order thought procedures." But there are others as well. Here belong also the techniques and materials commonly embraced by the term "sensory aids"—drawings, pictures, maps, films, diagrams, slides, solid articles intended primarily for classroom use, the geometric forms of architecture, and so on.

The argument for the abundant use of such aids is precisely the same as that for lower-order thought procedures. Learning is progressive in character. The abstractions of mathematics are not to be attained all at once, by some coordinated effort of mind and will. Instead, we must start with the child wherever he is, at the foot of the ladder, or at some point higher up. Well chosen sensory aids reveal the nature of the final abstractions in a way which makes sense to the child. If he can work out the new relationships in a concrete way and can himself test their validity in an objective setting, he has faith and confidence at the start; and he is the readier to learn with understanding the more abstract representations of mathematics. Sensory aids, like many so-called crutches, are then not only admissible under the conception of learning which I am outlining: they are obligatory.

On the other hand, the connectionistic view of learning does not predispose the teacher to employ temporary aids to the extent to which he should employ them. Instead, for reasons which I have already mentioned, it leads him too quickly to abstract practice or drill. And this is the third objection I have listed for consideration.

3. Faulty practice.—It would be false to accuse connectionists of spreading the gospel that "Practice makes perfect."[5] Nevertheless, the teacher who accepts their view of learning comes easily to rely upon drill as his exclusive or major teaching procedure. Listen again to the litany: Identify for the learner the situation to which he is to react, and the response he is to make to the situation; have him make the connection under satisfying conditions; have him exercise the connection until it is firmly established. Does not this *sound* like an exhortation to drill?

The apparently innocuous statement, "Practice makes perfect," is full of dynamite because it conceals important issues. Practice *does* make perfect in one sense of the words "practice" and "perfect"; but it makes for superficial learning in another sense of these same words. I can make my point by using a crude illustration.

Suppose that I, an amateur at golf, dub my drive: the ball barely trickles off the tee. What do I do? Practice? If so, in what sense? Well, if I practice in the sense of *repeating* my act, I grasp the club the same way, stand the same way, and swing the same way. With what result? I shall certainly get the same result—with this difference: I shall have become a bit more proficient in my poor drive. That is to say, there will be a little more economy and ease in my movements; I shall be more certain of what will happen; I may swing a little more quickly and precisely. Now, suppose I continue my repetitive practice over a period of time. Eventually, I shall become the most efficient poor golfer on the course, for repetitive practice will have brought its consequences.

But, you say, no golfer would do anything so silly as this; he would know what would happen. At first his practice would not be repetitive, but varied. That is to say, he would try to *avoid* the first combination of movements which produced the poor drive. He would stand differently, hold the club differently, swing differently, and so on. Only when he had arrived at the best possible combination of movements, as judged by the kind of drive it produced, would he start repetitive practice. And he would repeat purposely, because he would know that repetition could now give him the efficiency he desired.

But you are teachers of mathematics. What does this little excursion into golf have to do with the teaching of mathematics? At the risk of being dogmatic, I should say a good deal. As a matter of fact, we learn motor skills (the drive in golf) as we learn abstractions (mathematics). It is only because we cannot directly observe the activities in ideational learning that we come to think them different for motor learning. However, the appearance of difference vanishes when we adopt appropriate techniques for observation. Then we see the essential sameness in the case of motor skills and of abstractions. In both instances learning is characterized by the organization of behavior at successively higher levels. It is because of this essential sameness that the illustration from golf may prove helpful in showing the real function of practice in learning mathematics.

> The apparently innocuous statement, "Practice makes perfect," is full of dynamite because it conceals important issues

We are now ready to formulate two statements which should supplant the familiar "Practice makes perfect." (a) "*Varied* practice leads to the discovery of the right combination of movements and ideas." (b) "*Repetitive* practice produces efficiency, but at whatever level of performance the learner has attained." The proviso in the second statement is highly important: if repetitive practice is introduced prematurely, the learner is "frozen" at his level of performance. He steadily becomes more proficient at an undesirably low level of maturity. Repetitive practice cannot move the learner to a higher or more mature level. If, under conditions of drill or repetitive practice, the learner does actually move on to a higher level, the credit does not belong to the drill to which he has been subjected. An examination into his behavior will reveal that he has deserted the prescribed repetitive practice and has struck out for himself into varied practice.

It follows from what I have just said that in the end we defeat our own purposes by introducing repetitive practice too soon. In the early stages of mathematical learning we need to institute activities which will enable the learner to explore understandingly the new area which he is entering. The learner is not exploring this area when he does nothing but repeat what he has been told or solve problems in the way in which he has been shown.

Continued practice of this kind can yield nothing more than superficial learning: his efficiency may mislead us into thinking that he has a more thorough grasp than he actually has.

It also follows from what I have said that there is plenty of room and a great deal of need for repetitive practice in mathematics. Extremists in education who have reacted quite properly against premature drill do not correct the evil when they say that drill has no place at all in modern teaching technique. Such individuals would certainly engage in repetitive practice in golf; they would not for a minute believe that the limited insight of one lucky drive would give them command of the stroke. Yet, they would abolish drill in ideational learning. But drill is an inappropriate teaching procedure only when it is called upon to do what it cannot do. It is entirely appropriate when the goal is efficiency.

4. Remedial teaching.—We come now to the fourth objection to the prevalent conception or misconception of learning: uncritical acceptance of connectionism is likely to lead to the misinterpretation of error and to the use of inappropriate remedial measures.

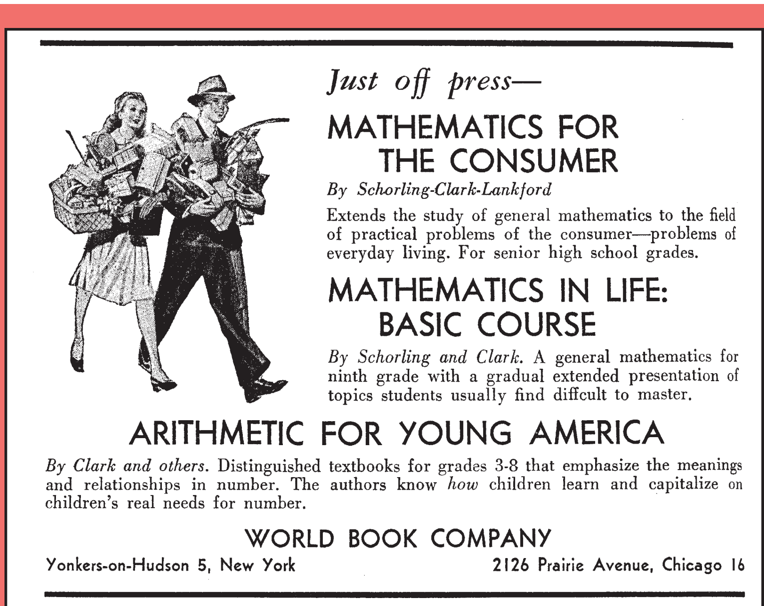
When a child makes a mistake, it is easy, but misleading, to say that he has made the wrong connection or that he has failed to make the correct one. And it is easy, but wrong, to assume that that mistake can be remedied merely by showing the child the correct response and having him practice until a connection is formed. This analysis of error and the proposed remedy might suffice if the learner were dealing with nonsense syllables or with puzzles which he could solve only by accident.[6] If the learner fails to recall one of the syllables or if he makes the wrong turn, in a maze, we give him the correct syllable, or we tell him his error in the maze or show him the correct turning. Remedial instruction in such learning assignments seems to be simple indeed: tell or show, and practice.

But the classroom should present exceedingly few learning tasks of these kinds, particularly in mathematics. Of course the learner must memorize arbitrarily predetermined characters and symbols, and in this case remedial instruction consists primarily in telling or showing and then in practicing. But, typically, learning tasks in mathematics are far more complex since they involve meanings and understandings.[7] I believe you will agree that most errors in mathematics are the result, not of imperfectly learned symbols, but of incomplete understandings, of inappropriate thought processes, and of faulty procedures.

Consistently throughout this paper I have stressed the progressive character of learning: the learner moves from level to level in thought processes, each successive level being more mature, more abstract, more adult-like than the preceding. Except in the case of imperfect mastery—and drill is then the remedial measure—except in such cases errors come from failure to traverse the stages and levels of thinking in an orderly fashion. Called upon to perform at a level higher than any he has yet attained and given no guidance to reach the higher level, the child has but three courses of action open to him. (a) He can refuse to learn. Refusal may take several different forms. One form is, “I won’t.” Under ordinary conditions of schooling this form is not common. Another form is, “I can’t”; a third is, “I don’t want to,” or “I don’t care.” The result of refusal, by whatever form, is indifference toward mathematics or dislike of it, which may be accompanied by wide-spread feelings of frustration.

If the child does not refuse to learn, he may adopt a second method of extricating himself from his predicament. (b) He may do his best to perform as he is asked to perform. This method is probably the commonest, and it reflects the effects of years of training in docility. Of course the child cannot actually do what he is supposed to do, but by blindly following rules and by profiting from model solutions he may *seem* for a time to be successful. His answers are correct, and he gets them with reasonable promptness. This evidence of learning, based upon the criteria of rate and accuracy, is spurious. The skill which is developed will not be useful except in the situations in which it is learned, and after a short while even this degree of skill is gone.

(c) The child’s third method of surmounting his difficulty is to fool his teacher, to continue actually to perform at his attained (but not at the expected) level, but to conceal this fact. This method is suc-



Sample ad, *Mathematics Teacher* 40 (March 1947) cessful when the child develops enough proficiency with his lower-order procedure to equal the performance of other children who are higher in the scale of maturity. And it is not impossible to attain this degree of proficiency. Many children do, and the fact is not discovered until later, perhaps a year or two, when with the growing complexity of learning tasks the low-order procedures are no longer effective.

It should be clear that no one remedial technique will be successful with these different types of disability. The child whose attitude toward mathematics has been ruined needs to have that attitude corrected. The working of masses of unenlightened and unenlightening examples and problems will not reach the source of difficulty. If the undesirable attitude arose because of inability to understand and of a consequent series of failures, the child's attitude will improve only when he understands and when he has had ample experiences of a successful kind.

Similar treatment may be necessary for the child who has developed spurious skills, only to lose them. More drill will serve only to revive for a time the skills which never were worth the trouble to cultivate. It is for this reason that so-called refresher courses in mathematics in the high school must frequently fail of their object. What most high-school students need is not more of the kind of instruction which produced their deficiencies, but a type of instruction of which they have not had enough. If the giving of this instruction means that the usual systematic mathematics courses in the high school must be sacrificed for many children, what of it? With the appallingly weak foundations many high-school students have in earlier phases of mathematics, they can hardly profit from the more advanced phases to which they have been traditionally exposed.

I have considerable sympathy for the child who has adopted the third way out of difficulty, that of remaining at a low level of performance and of developing proficiency therewith. After all, whatever he does, and however far he may be from the desired level of performance, he knows what he is about. In this connection let me recall the case of George who had trouble with bridging the decades in addition. The way to secure a higher level of thought process in such cases is certainly not to deny the child knowledge about that higher process, at the same time forbidding him the only process he has. Nor is it any more effective to assign large bodies of drill, for, in the case of less honorable children than George, the drill examples are but so many more invitations to increase facility of performance at a low level. Here as elsewhere, remedial measures must accord with some particular kind of deficiency.

I cannot quit this matter of remedial instruction without commenting upon certain practices which I have witnessed. I am both amused and irritated by the behavior of some college professors of mathematics when their weaker students come to them for help. Observe them, or observe a typical pair. The professor greets the confused student pleasantly; he sits down with him at a table, turns to the section of the text which is causing trouble, and neatly copies an example on a nice, clean sheet of slick white paper. Then they go to work. But before we come to the "work," let us note that the professor copies the example with a *pen* and he does all his work with a *pen.* Now perhaps memories of my own difficulties in college mathematics make me hypersensitive, but I resent that pen! I claim that the pen adds insult to injury. Consider the gulf its use establishes between student and professor. The student does his mathematical work with a pencil which is equipped with a large, competent eraser—and he uses that eraser frequently and vigorously. But the professor! he is so sure of himself that he can use a pen, thereby making a record which cannot easily be altered—only of course the professor knows that he won't have to alter his record. So does the student, and this knowledge may shake still more his already wavering self-confidence.

But drill is an inappropriate teaching procedure only when it is called upon to do what it cannot do

But I have admitted that my prejudice against the pen may be purely personal. So, let us get back to the work. What happens? The professor writes out each step in the solution calmly and certainly, meanwhile accompanying himself with a monologue. I cannot say that he is carrying on a conversation because the student contributes no verbal comment. Indeed, it is to be questioned whether he contributes anything at all, including understanding of what is going on. At the conclusion of the exhibition the professor settles back, well pleased with himself, and says mildly, "Well, there it is. Do you see how to do it?" Courtesy alone would require the student to say that he did see it, even if the stupefaction which magic produces did not render him incapable of more than nodding his head weakly or saying merely, "Yes."

Whatever name we give to this séance, we cannot call it remedial instruction. It might be a good idea to deny the professor all use of pencil and paper—and certainly of pen and paper! The student has come for *help,* not for a demonstration of the professor's skill in mathematics. If the professor were unable to write out his own processes, he might give the student the kind of help he has come for. The student, not the professor, should do the work. He should go as far as he can on his own; when he can go no further, he should be questioned and guided through questions to locate his difficulty and to analyze its nature. Through continued questioning he should be led to suggest possible next steps and then to evaluate these steps himself. But at all stages the student should be required to make use of his own knowledge (to the extent that he has any) and he should be allowed to identify his deficiencies himself and to feel that he is making progress by his own efforts. Remedial instruction of this kind is worthy of the name, and the results justify the time and energy that must be expended to secure them.

So much for the four instructional weaknesses which I listed for discussion, weaknesses which, if they cannot be attributed to the connectionistic view of learning, have certainly not been dispelled by the general acceptance of this view in American education. At the outset I promised that my comments would be constructive as well as critical. I think I have kept my promise. For a conception of learning which may be helpful so long as we deal with the most uncomplicated types of learning I have offered a substitute which may be more helpful for the kinds of learning which are involved in mathematics.

This latter conception stresses the notion of progressive reorganization. It emphasizes the essential continuity of learning. It points out that the learning of relationships and the development of meanings take time which is filled with suitable activities. It defines the kinds of practice which are appropriate at different stages in learning. It guarantees that children will not be hurried toward empty verbalizations but will be directed toward useful abstractions. In a word, it is a conception which provides insights into the course of learning which children must pursue if they are to attain the approved objectives of mathematics. I commend this conception to you teachers who are charged with the responsibility of guiding children to a meaningful and intelligent grasp of mathematics.

ENDNOTES

1. Paper read before the Regional Meeting of The National Council of Teachers of Mathematics in Detroit, February 19, 1944.

2. I have never been able to find much that is *wrong* (demonstrably unsound) in connectionism. Most of the direct attacks, theoretical and experimental, seem to me to be rather futile: they have failed to show that learning is anything other than the formation of connections (if the term "connection" be interpreted as broadly as connectionists interpret it). As a matter of fact, I suspect that, neurologically at least, something very much like the processes suggested by connectionists actually occurs in learning (this, in spite of the research of Lashley and others). Objection here is raised to connectionism purely on the ground, as stated above, that it has not been helpful in the practical business of educating children. Perhaps an analogy is in order. The layman reads that all matter is reducible, according to modern physics, to electrons, neutrons, etc., in a word, to non-matter. He can accept this view of matter as a fact. At the same time, acceptance of this view does not in any way affect the manner in which he deals with matter, however essential this view is to the work of the research physicist.

3. Of all the exponents of connectionism Gates has argued most cogently in this vein. Arthur I. Gates, "Connectionism: Present Concepts and Interpretations." *The Psychology of Learning,* Chapter IV. Forty-first Yearbook of the National Society for the Study of Education, Part II. Bloomington, Ill.: The Public School Publishing Co., 1942.

4. In the literature of connectionism I can find statements which contradict all these charges. That is to say, connectionists recognize, in theory, the evils which are listed above. But the habit of thinking of learning as connection-forming almost inevitably oversimplifies learning and affects teaching adversely. Perhaps no better example of this tendency is to be found than in the books *The Psychology of Arithmetic* and *The Psychology of Algebra,* both written by the originator of connectionism, Professor E. L. Thorndike.

5. Some connectionistic accounts of learning contain explicit warnings against over-reliance upon repetitive practice.

6. As is well known, the connectionist *as a connectionist* (that is, when he is concerned primarily with theory) has experimented chiefly with just such experimental problems. Even in such learning situations as he sets for the learner the connectionist would, if he but attended to what the learner does (his process), discover that learning is much more complicated than it appears in his account in terms of rate and accuracy. He would discover, even in the case of nonsense syllables and of mazes, that the learner approaches the final goal of efficiency through a process of reorganization at successively higher levels. Gates has recognized this fact to a far greater extent than have most connectionists. See: Arthur I. Gates, *Psychology for Students of Education,* Revised Edition. New York: The Macmillan Co., 1930. Chapter XI.

7. The connectionist of course agrees, but he has what to him is an explanation of the complexity. The learner is merely forming many connections at the same time, concomitantly and in succession. I can agree that this may possibly be so. At the same time I find little in this view which provides guidance to the teacher. ∞